# Math 249 Lecture 29 Notes

Daniel Raban

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## 1 Cycle Index Multiplication and Composition

### 1.1 Relationship to plethystic substitution

Last time we proved the following proposition.

**Proposition 1.1.** *Let $F$ be a species and $\mathcal{A}$ be a weighted set with ordinary generating function $A$, then $Z_F[A]$ is the ordinary generating function for unlabeled $F$-structures decorated with $\mathcal{A}$.*

**Corollary 1.1.** *If $F, G$ are species,*

$$Z_{FG} = Z_F Z_G.$$

*Proof.* It is enough to show that for every $A$,

$$Z_{FG}[A] = Z_F[A] Z_G[A].$$

A decorated $FG$-structure will be a decorated $F$-structure on one part of the set and a decorated $G$-structure on the rest. $\square$

**Definition 1.1.** Let $f, g \in \Lambda$. Then we define *plethystic composition* as

$$f * g := f|_{p_k \mapsto (p_k * g)}, \quad \text{where} \quad p_k * g = g|_{p_\ell \mapsto p_{k\ell}}.$$

Notice that $p_k[p_\ell[A]] = p_{k\ell}[A]$, so

$$(f * g)[A] = f[g[A]].$$

We also get the following corollary.

**Corollary 1.2.** *If $F, G$ are species,*

$$Z_{F \circ G} = Z_F * Z_G.$$

*Proof.* It is enough to show that for every $A$,

$$Z_{F\circ G}[A] = Z_F[Z_G[A]].$$

A decorated $F \circ G$-structure will be an $F$-structure on blocks partitioning a set and a decorated $G$-structure on each block. $\square$

**Remark 1.1.** Recall that the Schur functions correspond to the irreducible characters of $S_n$ via the Frobenius characteristic map. Viewing the cycle index in terms of the Frobenius map, the multiplication rule gives us a way to view induced representations of $S_k \times S_\ell \subseteq S_{k+\ell}$ up to $S_{k+\ell}$.

Similarly, recall that the Schur functions correspond to the irreducible characters of $\mathrm{GL}_n(\mathbb{C})$. An action $GL_n(\mathbb{C}) \circlearrowright \mathbb{C}^m$, gives us a homomorphism $GL_n(\mathbb{C}) \to GL_m(\mathbb{C})$, and so viewing the cycle index in terms of the Frobenius map defines a composition of such homomorphisms.

## 1.2 Examples and computer-aided computation

**Example 1.1.** let $T$ be the species of rooted trees. We had the species isomorphism

$$T = X_1(E \circ T).$$

Then we have that

$$Z_t = Z_X(Z_E * Z_T).$$

We know that

$$Z_E = \Omega = 1 + h_1 + h_2 + \cdots = \exp \sum_{k=1}^{\infty} \frac{p_k}{k},$$

$$Z_{X_1} = p_1,$$

so we get the recurrence

$$Z_T = p_1(\Omega * Z_T) = p_1 \exp \sum_{k=1}^{\infty} \frac{1}{k}(p_k * Z_T).$$

This gives us that

$$U_T(x) = Z_T[x] = x\Omega[Z_T[x]] = x \exp \sum_{k=1}^{\infty} \frac{1}{k} U_T(x^k),$$

which was something Polya proved using a more ad-hoc method; this was a fact that predated this theory.

Let's successively approximate $Z_T$.

$$0 = 0 + O(x)$$

$$x\exp(0) = x + O(x^2)$$

$$x\exp(x + \cdots) = x(1 + x + \cdots) = x + x^2 + \cdots$$

$$\begin{aligned} x\exp(x + x^2 + \cdots + \frac{1}{2}(x^2 + \cdots) + \cdots) &= x\exp(x + \frac{3}{2}x^2 + \cdots) \\ &= x\exp(x)\exp(\frac{3}{2}x^2 \cdots) \\ &= x(1 + x + \frac{x^2}{2} + \cdots)(1 + \frac{3}{2}x^2 + \cdots) \\ &= x(1 + x + \frac{x^2}{2} + \frac{3}{2}x^2 + \cdots) \\ &= x + x^2 + 2x^3 + \cdots \end{aligned}$$

**Example 1.2.** Let $G$ be the species of graphs, and consider a weighted cycle index, where we weight by $q^{|E(G)|}$:

$$Z_G(p_1, p_2, \ldots; q) = \sum_n \frac{1}{n!} \sum_{\sigma \in S_n} |G(n)^\sigma| \, p_{\gamma(\sigma)}$$

We can compute this with a computer to get that the generating function for unlabeled graphs is

$$\begin{aligned} &1 + x + (1+q)x^2 + (1 + q + q^2 + q^3)x^3 \\ &\quad + (1 + q + 2q^2 + 3q^2 + 2q^4 + q^5 + q^6)x^4 \\ &\quad + (1 + q + 2q^2 + 4q^3 + 6q^5 + 6q^6 + 4q^7 + 2q^8 q^9 q^{10})x^5 + \cdots \end{aligned}$$

Every graph is a union of connected graphs, so we have that $G \cong E \circ G_c$. We can the compute the generating function for connected unlabelled graphs with a computer, as well.